# The transitivity of Conway's $M_13$(Theory and Applications of Combinatorial Designs with Related Field)



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# The transitivity of Conway's $M_{13}$

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## 1 Introduction

Mathieu groups $M_{11}, M_{12}, M_{23}$ and $M_{24}$ are the only nontrivial 4-transitive permutation groups, and $M_{12}, M_{24}$ are the only nontrivial 5-transitive permutation groups. Conway introduced a set of permutations $M_{13}$ on 13 letters, which contains Mathieu group $M_{12}$, and he claims that $M_{13}$ is 6-transitive in some sense.

Martin and Sagan [2] generalized the concept of transitivity for a set of permutations. For a partition $\lambda$ of a positive integer $n$, a subset $D$ of the symmetric group $S_n$ is said to be $\lambda$-transitive if there exists $r>0$ such that for any partitions $P, Q$ of shape $\lambda$, $\sharp\{\tau\in D \mid P^{\tau}=Q\}=r$. In particular, $D$ is $(n-t, 1, \ldots, 1)$-transitive if and only if $D$ is $t$-transitive.

Conway's $M_{13}$ is not $(7,1,1,1,1,1,1)$-transitive according to this definition, so Martin and Sagan raised a question to determine the full transitivity of $M_{13}$.

In this paper, we give a recursive definition of the elements of $M_{13}$, and answer the question of Martin and Sagan.

## 2 Construction of $M_{13}$

Let $\Omega := \{0, 1, \ldots, 11, \infty\}$ be the set of points of a projective plane $P$ of order 3, and $\Omega_{12} := \Omega - \{\infty\}$. For $a \in \Omega_{12}$ we define a permutation on $\Omega$ by $\sigma(a) := (\infty\ a)(b\ c)$ where $\{\infty, a, b, c\}$ is the line of $P$, determined by $a, \infty$, and set $\sigma(\emptyset) = \sigma(\infty) = id_{\Omega}$. Recursively, for an integer $k$ such that $k \geq 2$ and $a_1, a_2, \ldots, a_k \in \Omega$ such that $a_1 \neq a_2 \neq \cdots \neq a_k$, define

$$\sigma(a_1, a_2, \ldots, a_k) := \tau(\infty\ a_k^{\tau})(b^{\tau}\ c^{\tau})$$

where $\tau = \sigma(a_1, \ldots, a_{k-1})$ and $\{a_{k-1}, a_k, b, c\}$ is the line determined by $a_{k-1}, a_k$. Note that $\sigma(a)$ is the move $a|bc$, and a *triangular permutation* in the sense of [1] is of the form $\sigma(a, b, \infty)$. The sets $M_{13}, M_{12}$ are defined as

$$M_{13} := \{\sigma(a_1, \ldots, a_k) \mid k \in \mathbb{N},\ a_i \in \Omega,\ a_i \neq a_{i+1}\ (1 \leq i \leq k-1)\},$$
$$M_{12} := \{\tau \in M_{13} \mid \infty^{\tau} = \infty\}.$$

The next proposition is useful to describe the elements of $M_{13}$.

**Proposition 1.** *Let $a_1, \dots, a_k, b_1, \dots, b_l \in \Omega$ be such that $a_1 \neq \dots \neq a_k \neq \infty \neq b_1 \neq \dots \neq b_l$. Then*

$$\sigma(a_1, \dots, a_k, \infty, b_1, \dots, b_l) = \sigma(b_1, \dots, b_l) \cdot \sigma(a_1, \dots, a_k, \infty).$$

*Proof.* We prove by induction on $l$. Let

$$\begin{aligned} \rho &= \sigma(b_1, \dots, b_{l-1}), \\ \pi &= \sigma(a_1, \dots, a_k, \infty), \end{aligned}$$

so $\sigma(a_1, \dots, a_k, \infty, b_1, \dots, b_{l-1}) = \rho\pi$ by the inductive hypothesis. Suppose that the line determined by $b_{l-1}, b_l$ is $\{b_{l-1}, b_l, c, d\}$. Then

$$\begin{aligned} \sigma(a_1, \dots, a_k, \infty, b_1, \dots, b_l) &= \rho\pi(\infty\ b_l^{\rho\pi})(c^{\rho\pi}\ d^{\rho\pi}) \\ &= \sigma(b_1, \dots, b_l)(\infty\ b_l^{\rho})(c^{\rho}\ d^{\rho})\pi(\infty\ b_l^{\rho\pi})(c^{\rho\pi}\ d^{\rho\pi}) \\ &= \sigma(b_1, \dots, b_l)\pi(\infty^{\pi}\ b_l^{\rho\pi})(c^{\rho\pi}\ d^{\rho\pi})(\infty\ b_l^{\rho\pi})(c^{\tau}\ d^{\rho\pi}) \\ &= \sigma(b_1, \dots, b_l) \cdot \sigma(a_1, \dots, a_k, \infty). \end{aligned}$$

□

The following propositions are obvious.

**Proposition 2.** *If $i$ is an integer such that $1 \leq i \leq k$ and $x \in \Omega - \{a_i\}$, then*

$$\sigma(a_1, \dots, a_k) = \sigma(a_1, \dots, a_i, x, a_i, a_{i+1}, \dots, a_k).$$

**Proposition 3.** *For $a, b \in \Omega_{12}$ such that $\{a, b, \infty\}$ is contained in a line,*

$$\sigma(a, \infty) = \sigma(a, b, \infty) = id_\Omega.$$

With these propositions, we prove the following theorem.

**Theorem 4.** *$M_{12}$ is the group generated by triangular permutations, and*

$$M_{13} = \coprod_{a \in \Omega} \sigma(a) M_{12}.$$

*Proof.* Let $\alpha = \sigma(a_1, \dots, a_k)$. Then $\alpha \in M_{12}$ if and only if $a_k = \infty$. For $i$ in $\{1, \dots, k-1\}$, if $a_i \neq \infty$ then we insert $\infty, a_i$ between $a_i$ and $a_{i+1}$ by Proposition 2. So by Propositions 1 and 3, $\alpha$ is written as a product of triangular permutations.

If $a_k = \infty$, then $\alpha \in M_{12}$. Otherwise, Proposition 2 implies

$$\alpha = \sigma(a_1, \dots, a_k, \infty, a_k)$$

so $\alpha \in \sigma(a_k) M_{12}$ by Proposition 1. □

# 3 Transitivity of $M_{13}$

An integer tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ is called a partition of a positive integer $n$ if $\lambda_i \geq \lambda_{i+1} \geq 0$ and $\sum_{i=1}^{k} \lambda_i = n$. A partition $P = (P_1, \ldots, P_k)$ of the set $\Omega_n := \{1, \ldots, n\}$ is said to have shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n$, if $|P_i| = \lambda_i$.

**Definition 5.** Let $n$ be an integer and $D$ be a set of permutations on $\Omega_n$. For a partition $\lambda$ of $n$, we say that $D$ is $\lambda$-transitive if there exists $r > 0$ such that for any partitions $P, Q$ of shape $\lambda$, $\sharp\{\tau \in D \mid P^\tau = Q\} = r$.

For example, a permutation group $G$ is $t$-transitive on $\Omega_n$ if and only if $G$ is $(n-t, 1^t)$-transitive, where $1^t$ means $\underbrace{1, \ldots, 1}_{t}$. We first prove the following general result.

**Lemma 6.** *For each $i \in \Omega_{n+1} = \{1, \ldots, n+1\}$, let $a_i$ be a permutation on $\Omega_{n+1}$ such that $i^{a_i} = n+1$, and $G$ be a permutation group on $\Omega_n = \{1, \ldots, n\}$. If $G$ is $(n-t, 1^t)$-transitive on $\Omega_n$ and*

$$D = \bigcup_{i \in \Omega_{n+1}} a_i G,$$

*then $D$ is a $(n-t+1, 1^t)$-transitive set on $\Omega_{n+1}$.*

*Proof.* For $t$-tuples $X = (x_1, \ldots, x_t), Y = (y_1, \ldots, y_t)$ of distinct elements of $\Omega_{n+1}$, we define

$$D_Y^X := \{\tau \in D \mid X^\tau = Y\}.$$

First, we suppose that $Y$ contains $n+1$, for example, $y_1 = n+1$. Then $D_Y^X \subset a_{x_1} G$, and $\{x_k^{a_{x_1}} \mid 2 \leq k \leq t\}, \{y_k \mid 2 \leq k \leq t\} \subset \Omega_n$. By the $(n-(t-1), 1^{t-1})$-transitivity of $G$ on $\Omega_n$,

$$\begin{aligned} |D_Y^X| &= \sharp\{g \in G \mid (x_k^{a_{x_1}})^g = y_k \ (2 \leq k \leq t)\} \\ &= \frac{|G|}{n \cdot (n-1) \cdots (n-(t-2))}. \end{aligned}$$

Next, we assume that $n+1$ does not appear in $Y$. For an integer $i$ such that $1 \leq i \leq t$, if $a_i g \in D_Y^X \cup a_i G$ then $i \notin X$ and $\{x_1^{a_i}, \ldots, x_t^{a_i}\}, Y \subset \Omega_n$, so by the $(n-t, 1^t)$-transitivity of $G$ on $\Omega_n$,

$$\begin{aligned} |D_Y^X| &= \sum_{i \in \Omega_{n+1} - X} \sharp\{g \in G \mid (X^{a_i})^g = Y\} \\ &= |\Omega_{n+1} - X| \cdot \frac{|G|}{n \cdot (n-1) \cdots (n-(t-1))} \\ &= \frac{|G|}{n \cdot (n-1) \cdots (n-(t-2))}. \end{aligned}$$

□

By this lemma, we see that $M_{13}$ is $(8, 1^5)$-transitive. We will show that $M_{13}$ is not $(7, 6)$-transitive.

If $M_{13}$ is $(7, 6)$-transitive, then for any 6-element sets $P, Q$ of $\Omega_{13}$,

$$\sharp\{\tau \in M_{13} \mid P^\tau = Q\} = \frac{|G|}{\binom{13}{6}} = 720.$$

It is known that $M_{12}$ leaves the set of hexads invariant (see [1] for details). We define $H := \{h^{\sigma(a)} \mid a \in \Omega,\ h : \text{hexad}\}$. If $h = \{1, 2, 3, 4, 5, 6\}$ then $h^{\sigma(7)} = h^{\sigma(8)}$ and

$$|H| < |\Omega| \cdot 132 = \binom{13}{6},$$

so there is a 6-element set $P$ that is not contained in $H$. Taking $Q$ as a hexad, we obtain

$$\sharp\{\tau \in M_{13} \mid P^\tau = Q\} = 0.$$

Therefore $M_{13}$ is not $(7, 6)$-transitive.

We need to introduce the dominance order on partitions of $n$, in order to state a result of Martin and Sagan [2]. For two integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k), \mu = (\mu_1, \ldots, \mu_l)$, we define

$$\lambda \trianglelefteq \mu \iff \sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i, \text{ for any positive integer } j$$

where $\lambda_i = 0$ for $i \geq k$ and $\mu_i = 0$ for $i \geq l$.

**Theorem 7 (Martin and Sagan [2]).** *If a set $D$ is $\lambda$-transitive and $\lambda \trianglelefteq \mu$, then $D$ is $\mu$-transitive.*

Using this theorem, we can determine the transitivity of $M_{13}$.

**Theorem 8.** *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of 13. Then $M_{13}$ is $\lambda$-transitive if and only if $\lambda_1 \geq 8$.*

*Proof.* If $\lambda_1 \geq 8$ then $\lambda \trianglerighteq (8, 1^5)$. Since $M_{13}$ is $(8, 1^5)$-transitive, $M_{13}$ is also $\lambda$-transitive by Theorem 7. Suppose $M_{13}$ is $\lambda$-transitive for some $\lambda$ with $\lambda_1 \leq 7$. Then $M_{13}$ is $(7, 6)$-transitive by Theorem 7 again since $\lambda \trianglelefteq (7, 6)$. This is a contradiction. □

# References

[1] J. H. Conway, $M_{13}$, Surveys in combinatorics, 1997 (London), pp. 1–11, London Math. Soc. Lecture Note Ser., 241, Cambridge Univ. Press, Cambridge, 1997.

[2] W. J. Martin and B. E. Sagan, A new notion of transitivity for sets of permutations, to appear in J. London Math. Soc.